Limiting distribution of decoherent quantum random walks

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The behavior of one-dimensional quantum random walks is strikingly different from that of classical ones. However, when decoherence is involved, the limiting distributions take on many classical features over time. In this paper, we study the decoherence on both position and "coin" spaces of the particle. We propose an analytical approach to investigate these phenomena and obtain the generating functions which encode all the features of these walks. Specifically, from these generating functions, we find exact analytic expressions of several moments for the time and noise dependence of position. Moreover, the limiting position distributions of decoherent quantum random walks are shown to be Gaussian in an analytical manner. These results explicitly describe the relationship between the system and the level of decoherence.

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I. INTRODUCTION

Quantum random walks have recently gained great interest from physicists, computer scientists, and mathematicians. The interest was sparked by their important roles in developing highly efficient quantum algorithms. For instance, Grover's search algorithm [1] has time cost $O(\sqrt{N})$, in contrast to the ordinary search algorithm which has a cost of $O(N)$. This quantum search algorithm was proved to be closely related to the behavior of quantum random walks in [2,3]. As another example, Shor's algorithm also improved the speed of factorization dramatically [4]. The high efficiency of quantum algorithms is discussed in [5–7]. Experimental implementations of the algorithms are discussed in [8,9].

Besides their important applications, quantum random walks are very attractive due to their dramatic nonclassical behavior. After quantum random walks were defined in [10], many articles [11–15] studying the distribution of quantum random walks were presented. It is known that the observed nonclassical behavior is due to quantum coherence [16]. One of the most shocking differences [11] is that as time t grows, the variances of quantum random walks are $O(t^2)$ while the variances of classical random walks are $O(t)$. Various limit theorems of quantum random walks are established [12,13,17,18]. An excellent reference can be found in [19].

One of the most important issues surrounding the use of quantum random walks is that they are very sensitive to inevitable decoherence, which could be caused by many reasons, such as interactions with the environment and system imperfections [20–22]. The effect of decoherence is very important for the application of quantum algorithms, as discussed in [23,24]. For the one-dimensional case, in the model in [20], decoherence is introduced by measurements on the particle's chirality. Long-term first and second moments of the walk were obtained and numerical results showed that the distributions look like classical normal distributions. Similar results are found in other models [16,23,25–29]. In particular, all of the above papers mentioned the fact that the variance of the simulated position distribution grows linearly in time for large t when the quantum random walk is subject to decoherence.

These results stimulated us to prove that the position distribution of a one-dimensional decoherent quantum random walk normalized by $\sqrt{t}$, $P(\frac{x}{\sqrt{t}},t)$, converges to a normal distribution. Our work focuses on the one-dimensional discrete-time Hadamard walk with measurements taken on both position and chirality at each time step. This kind of decoherence is studied numerically in [8,23,25,26] but we will study it fully analytically.

We shall see that when the particle is not measured, then the system evolution is purely quantum and $P(\frac{x}{\sqrt{t}},t)$ does not converge. However, when the particle is measured subject to a small probability, $P(\frac{x}{\sqrt{t}},t)$ will converge to normal. In the limit, when the particle is measured at each step, then the system becomes purely classical and the normalized position distribution is asymptotically standard normal.

In the next section, we introduce the mathematical setup of decoherent quantum random walks. We then provide our methodology of generating functions and the decoherence equation. We next list our results and discuss the interesting phenomena that occur when p is small. Finally, we summarize and discuss our work. Mathematical proofs are given in the Appendixes.

II. DECOHERENT QUANTUM RANDOM WALKS

A. Pure quantum random walk system

We start with a brief description of the one-dimensional pure quantum random walk system. In the classical random walks, the particle moves to the right-hand or left-hand sides depending on the result of a coin toss. However, in the quantum random walks, the particle has its chirality {right, left} as another degree of freedom. At each step, a unitary transformation is applied to the chirality state of the particle and the particle moves according to its new chirality state.

We denote the position space of the particle by H_p, the complex Hilbert space spanned by the orthonormal basis $\{|x\rangle, x\in\mathbb{Z}\}$, where $\mathbb{Z}$ is the set of integers. We also denote the coin space by H_c as the complex Hilbert space spanned by the orthonormal basis $\{|l\rangle, l=1,2\}$ where 1 stands for "mov-

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ing right" and 2 stands for "moving left." The state space H of the particle is defined as

$$H = H_p \otimes H_c. \tag{1}$$

A vector $|\psi\rangle \in H$ with L^2-norm 1 is called a state and tells us the distribution of the particle's position and chirality upon measurements. The basis of H are denoted by $\{|x,l\rangle = |x\rangle \otimes |l\rangle : x \in \mathbb{Z}\},\ l=1,2\}$. Now we introduce the evolution operator which drives the particle. The shift operator $S:H \to H$ is defined by

$$S|x\rangle|l\rangle = \begin{cases} |x+1\rangle|1\rangle, & l=1, \\ |x-1\rangle|2\rangle, & l=2. \end{cases} \tag{2}$$

The coin operator $C:H_c \to H_c$ can be any unitary operator and is an analog to the coin flip in the classical walk. The evolution operator $U:H \to H$ is defined by

$$U = S(I_p \otimes C), \tag{3}$$

where I_p is the identity in the position space.

Now let $|\psi_0\rangle \in H$ be the initial state and let $|\psi_t\rangle = U^t|\psi_0\rangle$. The sequence $\{|\psi_t\rangle\}_0^\infty$ is called a one-dimensional quantum random walk.

The most famous and best-studied example of quantum random walks is the Hadamard walk, in which the coin operator is the 2×2 Hadamard matrix

$$H_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{4}$$

The quantum random walk associated with H_2 is called a one-dimensional Hadamard walk.

The probability of a particle at state $|\psi\rangle$ to be found at state $|\eta\rangle$ is defined by the norm squared of the inner product of $|\psi\rangle$ and $|\eta\rangle$, $|\langle\psi|\eta\rangle|^2$.

In particular, the probability of the quantum random walk, starting from the position $x=0$, with the coin in state m, to be found at x with coin state n is

$$W_{m,n}(x,t) = |\langle x,n|U^t|0,m\rangle|^2. \tag{5}$$

B. Decoherence

We focus on decoherence caused by measurements on both position and coin of the particle. A set of operators $\{A_i, i \in \mathbb{A}\}$ is called a measurement if

$$\sum_{i \in \mathbb{A}} A_i^* A_i = I, \tag{6}$$

where $\mathbb{A}$ is some index set and A^* is the adjoint operator of A, i.e., the complex conjugate of transposed matrix of A.

In this work, we consider the measurements in a similar manner as in [20]. Let p be a real number in [0,1], to denote the probability of the random walk being measured at each step. We define $A_c:H \to H$, such that $A_c = \sqrt{1-p}I$. Hence, the application of A_c represents the case of no measurement being made on the particle. We also let $A_{x,n}:H \to H$, so that $A_{x,n} = \sqrt{p}|x,n\rangle\langle x,n|$ is the decoherence projection to the subspace spanned by $|x,n\rangle$. Under this setup, the index set $\mathbb{A}$ is $\mathbb{A} = \{c\} \cup \{(x,n): x \in \mathbb{Z}, n=1,2\}$. Hence, by summing over the index set $\mathbb{A}$, we either apply the projective measurement operators $A_{x,n}$ with probability p or the identity operator A_c with probability $1-p$.

Let $|\psi\rangle$ be a state in H. Then the position distribution of the decoherent quantum random walk starting from $|\psi\rangle$, at time t is

$$P^\psi(x,t) = \sum_n \sum_{j_n \in \mathbb{A}} \cdots \sum_{j_1 \in \mathbb{A}} |\langle x,n|(A_{j_t}U)\cdots(A_{j_1}U)|\psi\rangle|^2. \tag{7}$$

In other words, the walk starts at $|\psi\rangle$, then we apply the evolution operator U and try to measure it. The process repeats until the tth step is finished. We then consider the position distribution of the particle. We call each $[j_1, j_2, \ldots, j_t, (x,n)]$ a path. We also call $\langle x,n|(A_{j_t}U)(A_{j_{t-1}}U)\cdots(A_{j_1}U)|\psi\rangle$ an amplitude function of the particle associated with the path. Many paths yield 0 amplitude due to the decoherence projections, the A_{xj}'s. However, the summation in Eq. (7) over all paths $[j_1, j_2, \ldots, j_t, (x,n)]$ gives the probability of observing the particle at position state $|x\rangle$ at time t.

At each step of a path, the walk is either not measured with probability $q=1-p$ or is measured at $|x,n\rangle$ with probability p. So when $p=0$, the walk is not measured and the system is the same as the pure quantum random walk previously defined. When $p=1$, the particle is interfered with at each step, hence the quantum behavior essentially disappears and the system is exactly classical.

We work on the decoherent Hadamard walk starting from position 0. We use

$$P_t(|\psi\rangle, |\phi\rangle) = \sum_{j_t \in \mathbb{A}} \ldots \sum_{j_1 \in \mathbb{A}} |\langle\phi|(A_{j_t}U)\ldots(A_{j_1}U)|\psi\rangle|^2 \tag{8}$$

to denote the probability, at time t, of a particle in the decoherent quantum random walk starting from $|\psi\rangle$ to be found at state $|\phi\rangle$. In particular, we denote the probability that at time t, the particle starting at $|0,m\rangle$ can be found at $|x,n\rangle$ by

$$P_{m,n}(x,t) = \sum_{j_1,\ldots,j_t \in \mathbb{A}} |\langle x,n|(A_{j_t}U)\cdots(A_{j_1}U)|0,m\rangle|^2. \tag{9}$$

Since we are interested in the limiting distribution of the walk, we focus on the Fourier transform of the above probabilities,

$$\hat{P}_{m,n}(k,t) = \sum_x P_{m,n}(x,t)e^{ikx}. \tag{10}$$

We consider two types of walks. We first consider the walk starting at the state $|\phi_0\rangle = \frac{1}{\sqrt{2}}|0,1\rangle + i\frac{1}{\sqrt{2}}|0,2\rangle$. We call this walk "symmetric" and denote its probability distribution by $P(x,t)$. Note that the characteristic function of the symmetric walk is

$$\hat{P}(k,t) = \frac{1}{2}\sum_m \sum_n \hat{P}_{m,n}(k,t). \tag{11}$$

From the above equation, we can see that its characteristic function is obtained by taking the average of those with initial chirality state m. Furthermore, in [11], it is shown that the pure quantum random walk starting with $|\phi_0\rangle$ has a symmetric position distribution. These are the reasons why we call it "symmetric."

We also consider the walk that starts at $|0,1\rangle$, i.e., the walk starting at 0 with chirality "right" and denote its probability distribution by $\tilde{P}(x,t)$. In this case, the characteristic function of this walk is

$$\hat{\tilde{P}}(k,t) = \sum_n \hat{P}_{1,n}(k,t). \quad (12)$$

Our goals are to show that as $t \to \infty$,

$$\hat{P}\left(\frac{k}{\sqrt{t}},t\right) \to e^{-(1/2)vk^2}, \quad (13)$$

for some positive number v in the symmetric walk case, as well as to show that as $t \to \infty$,

$$\hat{\tilde{P}}\left(\frac{k}{\sqrt{t}},t\right) \to e^{-(1/2)vk^2}, \quad (14)$$

in this specific initial state case.

III. GENERATING FUNCTIONS AND THE DECOHERENCE EQUATION

A. Generating functions

The direct calculation involves some formidable, very complicated combinatorics. Therefore, we introduce the idea of generating functions. The generating function of the decoherent quantum random walk is

$$P_{m,n}(x,z) = \sum_{t=0}^{\infty} P_{m,n}(x,t) z^t. \quad (15)$$

The Fourier transform of the generating function is

$$\hat{P}_{m,n}(k,z) = \sum_x P_{m,n}(x,z) e^{ikx}. \quad (16)$$

Note that for z in the unit disk $\{z : |z| < 1\}$, since $|\hat{P}_{m,n}(k,t)| \leq 1$ and $|P_{m,n}(x,t)| \leq 1$ for every t, the $\Sigma_{t=0}^{\infty} \hat{P}_{m,n}(k,t) z^t$ and $P_{m,n}(x,z)$ are analytic. Furthermore,

$$\sum_x \sum_{t=0}^{\infty} |P_{m,n}(x,t) e^{ikx} z^t|^2 < \infty. \quad (17)$$

Hence, by Fubini's theorem, we have

$$\hat{P}_{m,n}(k,z) = \sum_{t=0}^{\infty} \hat{P}_{m,n}(k,t) z^t, \quad (18)$$

i.e., $\hat{P}_{m,n}(k,z)$ is analytic and $\hat{P}_{m,n}(k,t)$ is the coefficient of z^t in the expansions of $\hat{P}_{m,n}(k,z)$. Therefore, instead of finding $\hat{P}_{m,n}(k,t)$ directly, we first find the explicit formulas of $\hat{P}_{m,n}(k,z)$ and then we apply Cauchy's theorem

$$\hat{P}_{m,n}(k,t) = \frac{1}{2\pi i} \oint_{|z|=r} \frac{\hat{P}_{m,n}(k,z)}{z^{t+1}} dz, \quad (19)$$

for some $0 < r < 1$, to obtain $\hat{P}_{m,n}(k,t)$.

B. Decoherence equation

The functions $\hat{Q}_{m,n}(k,z)$ and $Q_{m,n}(x,z)$ are very important in our proofs. We let $\hat{W}_{m,n}(k,t) = \Sigma_x W_{m,n}(x,t) e^{ikx}$ be the Fourier transform of the pure Hadamard walk. We also let $\hat{Q}_{m,n}(k,z) = \frac{p}{q} \Sigma_{t=1}^{\infty} \hat{W}_{m,n}(k,t)(qz)^t$ for $0 < p \leq 1$ and $q = 1-p$. Note that $|\hat{W}_{m,n}(k,t)| \leq 1$. Hence, for $z \in \{z : |z| < \frac{1}{q}\}$, $|\hat{Q}_{m,n}(k,z)| < \infty$. Therefore, $\hat{Q}_{m,n}(k,z)$ is analytic in $\{z : |z| < \frac{1}{q}\}$. Furthermore, let $Q_{m,n}(x,z) = \frac{p}{q} \Sigma_{t=1}^{\infty} W_{m,n}(x,t)(qz)^t$, and by Fubini's theorem again we have

$$\hat{Q}_{m,n}(k,z) = \frac{p}{q} \sum_{t=1}^{\infty} \hat{W}_{m,n}(k,t)(qz)^t = \sum_x Q_{m,n}(x,z) e^{ikx}. \quad (20)$$

Using the above notations, we derive the following theorem, whose proof can be found in the appendixes and in [30].

Theorem III.1 (decoherence equation). The functions $\hat{P}_{m,n}(k,z)$ are analytic in $\{z : |z| < 1\}$ and are meromorphic in $\{z : |z| < \frac{1}{q}\}$. Furthermore, if we denote the matrices of $[\hat{P}_{m,n}(k,z)]$ and $[\hat{Q}_{m,n}(k,z)]$ by P and Q, respectively, then

$$P = -\frac{q}{p} I + \frac{1}{p} (I - Q)^{-1}. \quad (21)$$

This equation establishes the relationship between the decoherent quantum random walk (left-hand side) and the pure quantum random walk (right-hand side). By working on the Fourier transform of the pure quantum random walk, we obtain the formulas of $\hat{P}_{m,n}(k,z)$ from this equation.

IV. MAIN RESULTS

We list our results for the two types of decoherent quantum random walks here. The step-by-step mathematical proofs of theorems in this section can be found in [30].

A. Results for the symmetric decoherent quantum random walk

The following theorem gives the closed form formula of the Fourier transform of the generating function of the walk. This formula synthesizes all the information of the walk and is crucial in proving our results.

Theorem IV.1. The Fourier transform of the generating function of the symmetric decoherent Hadamard walk, $\hat{P}(k,z)$, is given by

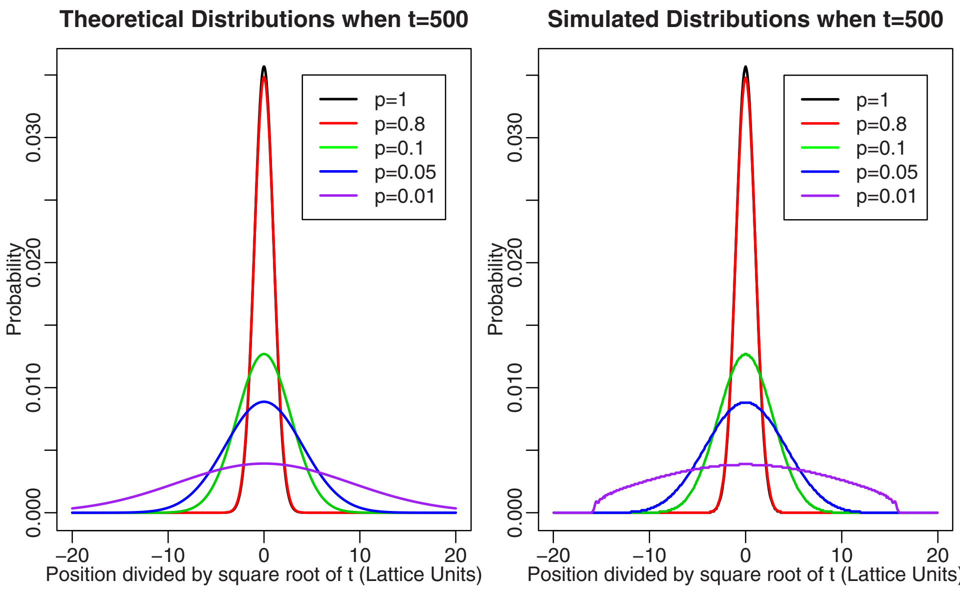

FIG. 1. (Color online) Comparison between the theoretical limiting position distributions and the simulated decoherent Hadamard walks when $t=500$. The legend refers to curves shown from top to bottom. p takes values of 1,0.8,0.1,0.05, and 0.01.

$$\hat{P}(k,z) = \frac{q(q-\cos^2 k)z^2 + p\cos kz + (1 - z\cos k)E}{pq\cos kz^3 - (pq+p)z^2 + p\cos kz + (z^2 - 2\cos kz + 1)E}, \tag{22}$$

where

$$E = \sqrt{[q^2z^2 - (1+\cos k)qz + 1][q^2z^2 + (1+\cos k)qz + 1]} \tag{23}$$

and the square root in the formula of E is defined through the principal logarithm.

We first show that the position distribution of the walk is symmetric with respect to the origin.

Theorem IV.2. Let μ_t be the expected position of the symmetric decoherent Hadamard walk on the line with $0<p\leq 1$ and $q=1-p$. Then $\mu_t=0$, $\forall\ t$.

We then consider the limiting distribution. We derive the following theorem for the limiting distribution of the symmetric decoherent Hadamard walk.

Theorem IV.3. For the symmetric decoherent Hadamard walk on the line with $0<p\leq 1$ and $q=1-p$, the characteristic function $\hat{P}(k,t)$ satisfies

$$\hat{P}\left(\frac{k}{\sqrt{t}},t\right) = \exp\left(-\frac{p+2\sqrt{1+q^2}-2}{2p}k^2\right) + O(t^{-1}) \tag{24}$$

as $t\to\infty$, i.e.,

$$P\left(\frac{x}{\sqrt{t}},t\right) \to N\left(0,\frac{p+2\sqrt{1+q^2}-2}{p}\right) \tag{25}$$

in distribution as $t\to\infty$.

This theorem states that after a long time, the position distribution of the particle is Gaussian. We see from the variance of the distribution that it is a mixture of the quantum and classical limiting distribution.

We give a plot of comparison for theoretical limiting distributions and simulated distributions with respect to different p's in Fig. 1. The probabilities in the theoretical distributions are obtained by integrating the densities from Eq. (25) over each interval of length $\frac{2}{\sqrt{500}}$.

We see that the numerical results compare very well with theoretical limits obtained in Theorem 3, except for $p=0.01$. This is because when $p=0.01$, $t=500$ is not large enough for the entire distribution to converge. However, even in this case, we can still see that a classical central peak of the distribution is forming.

We also see the trend of the distributions as p varies. When p increases to 1, the distribution is asymptotically standard normal, as in a classical walk. On the other hand, when p is decreasing to 0, the variance increases to infinity, meaning that $P(\frac{x}{\sqrt{t}},t)$ does not converge. In fact, in [12,13], it is shown that $P(\frac{x}{t},t)$ converges.

We also find the long-term variance of the symmetric walk as follows.

Theorem IV.4. For the symmetric decoherent Hadamard walk on the line with $0<p\leq 1$ and $q=1-p$, the variance $V(x,t)$ satisfies

$$V(x,t) = \frac{p+2\sqrt{1+q^2}-2}{p}t - \frac{2q^2}{p\sqrt{1+q^2}} - \frac{2}{p^2}(1+q^2-\sqrt{1+q^2}) + O(e^{-ct}), \tag{26}$$

for some $c>0$, as $t\to\infty$.

This theorem shows that for fixed p and large t, the standard deviation of the walk is growing linearly in $\sqrt{t}$. We plotted the standard deviations obtained from Eq. (26)

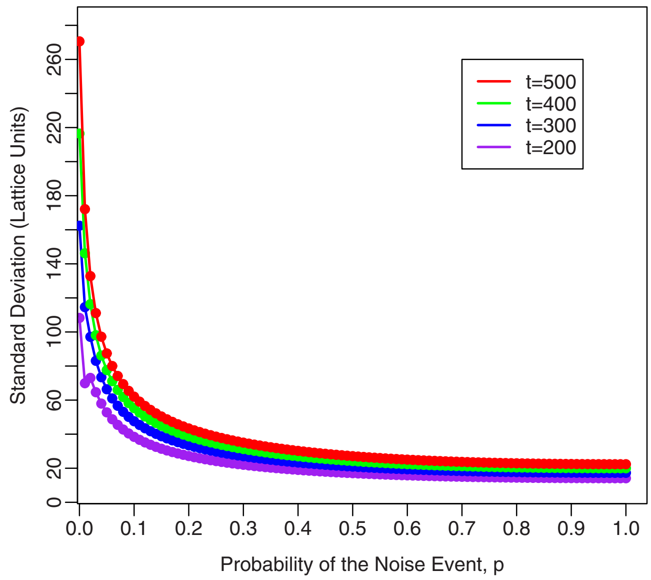

FIG. 2. (Color online) Standard deviation of the particle position as a function of p at $t=200$, 300, 400, and 500. The legend refers to curves shown from top to bottom. The grid of p is from 0 to 1 with increment 0.01.

against p's in Fig. 2. This picture compares very well with Fig. 1 in [26], where the values of standard deviations are from numerical simulations. Note that when $t=200$ and $p=0.01$, the standard deviation seems too low. This is because when $t=200$, the limiting phenomenon has not occurred yet for the walk with $p=0.01$. Therefore, the formula (26) is not a good approximation and the decoherent walk still resembles the pure quantum random walk. We discuss this in Sec. V.

B. Results for the decoherent Hadamard walk starting at $|0,1\rangle$

Now we consider the decoherent walk starting at $|0,1\rangle$. As before we first find the expected position of the walker at time t.

Theorem IV.5. For the decoherent Hadamard walk starting at $|0,1\rangle$ with $0<p\leq 1$ and $q=1-p$, if we let $\tilde{\mu}_t$ be the expected position at time t, then we have $\tilde{\mu}_t=\frac{\sqrt{1+q^2}-1}{p}+O(e^{-dt})$ for some $d>0$, as $t\rightarrow\infty$.

This theorem shows that the limiting expected position of the decoherent Hadamard walk is to the right-hand side of the origin, if the initial coin state is "right." We see that when $p\rightarrow 0$, $\tilde{\mu}_t\rightarrow\infty$. This finding is consistent with the result in [11] that the pure quantum random walk starting with chirality "right" drifts to the right-hand side.

For the second moment, we have the same result as for the symmetric walk.

Theorem IV.6. For the decoherent Hadamard walk starting at $|0,1\rangle$ with $0<p\leq 1$ and $q=1-p$, the variance $\tilde{V}(x,t)$,

$$\tilde{V}(x,t)=\frac{p+2\sqrt{1+q^2}-2}{p}t-\frac{2q^2}{p\sqrt{1+q^2}}-\frac{2}{p^2}(1+q^2-\sqrt{1+q^2})+O(e^{-ct}), \quad (27)$$

for some $c>0$, as $t\rightarrow\infty$.

Now we show that the limiting position distribution of the decoherent Hadamard walk starting at $|0,1\rangle$ is also Gaussian.

Theorem IV.7. For the decoherent Hadamard walk starting at $|0,1\rangle$ with $0<p\leq 1$ and $q=1-p$, the characteristic function $\hat{\tilde{P}}(k,t)$ satisfies

$$\hat{\tilde{P}}\left(\frac{k}{\sqrt{t}},t\right)=\exp\left(-\frac{p+2\sqrt{1+q^2}-2}{2p}k^2\right)+O(t^{-1/2}) \quad (28)$$

as $t\rightarrow\infty$, i.e.,

$$\tilde{P}\left(\frac{x-\tilde{\mu}_t}{\sqrt{t}},t\right)\rightarrow N\left(0,\frac{p+2\sqrt{1+q^2}-2}{p}\right) \quad (29)$$

in distribution as $t\rightarrow\infty$.

Remark IV.1. Note that here the converging speed is $O(t^{-1/2})$ while we have $O(t^{-1})$ for the symmetric walk. This is because when one takes the average of the $\hat{P}_{m,n}(\frac{k}{\sqrt{t}},t)$, the error terms in $O(t^{-1/2})$ cancel out one another. This result shows that the symmetric walk converges faster.

V. SPEED OF THE WALK WHEN p IS SMALL: PSEUDOQUANTUM PHENOMENON

In [11], it is shown that the long-term variance of the Hadamard walk is $(1-\frac{1}{\sqrt{2}})t^2$. We proved the same result via our approach by letting $p\rightarrow 0$.

Theorem V.1. Let $Q(x,t)$ denote the position distribution of the Hadamard walk on the line, the long-term variance $V_Q(x,t)$ satisfies $\frac{V_Q(x,t)}{t^2}\rightarrow(1-\frac{1}{\sqrt{2}})$ as $t\rightarrow\infty$.

We also investigated the case when p is small. Because current literature shows that the distribution of the pure Hadamard walk compares well with the uniform distribution over $[-\frac{t}{\sqrt{2}},\frac{t}{\sqrt{2}}]$, we also compare the decoherent Hadamard walk with it. Denote the uniform distribution over $[-\frac{t}{\sqrt{2}},\frac{t}{\sqrt{2}}]$ by U_t. We shall compare the variance of U_t and the variance of the symmetric walk. Note that the long-term variance of the decoherent walk is given by Eq. (26). Note also that the variance of U_t is given by

$$\mathrm{Var}(U_t)=\frac{t^2}{6}. \quad (30)$$

Hence, the difference $|\mathrm{Var}(U_t)-V(x,t)|$ is minimized at

$$t_0=\frac{6(\sqrt{1+q^2}-1)}{p}+3. \quad (31)$$

The minimizer p from Eq. (31) when $t_0=200$ is about 0.0124. This compares well with the numerical result in [26], which demonstrated that when $t_0=200$, the p that minimizes the difference between the symmetric walk and U_t is about 0.013. A plot of t_0 versus p is also given in Fig. 3.

The time t_0 is interesting in the sense that the variance of the decoherent walk could not be regarded as linear in t before t_0, i.e., the walk before t_0 could not be regarded as "classical." We call the period from 0 to t_0 "pseudoquantum" since the walk takes on quantum features. After t_0, the variance of the walk approaches the formula obtained in Eq.

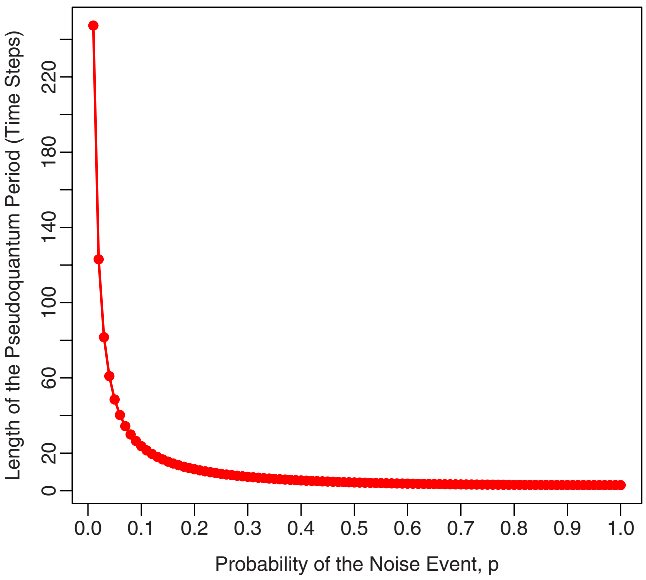

FIG. 3. (Color online) The pseudoquantum time t_0 as a function of p. The grid of p is from 0.01 to 1 with increment 0.01.

(26). For example, for $p=0.01$, $t_0=247.3$. Therefore, when $t=200$, the limiting behavior has not occurred and the decoherent Hadamard walk has more quantum features than classical ones. This explains why in Fig. 2, the standard deviation obtained from Eq. (26) is not accurate when $p=0.01$ and $t=200$.

VI. CONCLUSIONS

We have investigated the quantum walk with decoherence on both position and chirality states. Long-term limits are obtained for both the symmetric walk and the walk starting at 0 with chirality "right." We provide analytical explanations of the dynamics of the decoherent quantum walk system and we see that the system is indeed a mixture of quantum and classical ones. The limiting distributions of quantum random walks are shown to be Gaussian if decoherence occurs. These results are very important properties of the decoherent quantum random walks and could be essential for the development of quantum algorithms and experiments.

We also see that when p is small, the system remains nonclassical for a very long time. If a quantum algorithm can be finished before the classical features appear, then we may call it a "pseudoquantum" algorithm. However, we do not know how fast the "pseudoquantum" algorithms are as compared to the classical ones. Therefore, we suggest future studies on these areas.

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APPENDIX A: PROOF OF THEOREM III.1

We start with an observation about the decoherent quantum random walk and obtain a recursive formula. Then we apply that formula to the $\hat{P}_{m,n}(k,z)$ to establish the decoherence equation.

For any state $|\phi\rangle \in H$, $|\phi\rangle$ can be written as $|\phi\rangle = \Sigma_{y,l}\langle y,l|\phi\rangle|y,l\rangle$. By definition,

$$P_{t+1}(|0,m\rangle,|\phi\rangle) = qP_t(|0,m\rangle,U^*|\phi\rangle) + p\sum_{y,l}|\langle y,l|\phi\rangle|^2 P_t(|0,m\rangle,U^*|y,l\rangle). \tag{A1}$$

In particular, for $|\phi\rangle=|x,n\rangle$, we have

$$P_{t+1}(|0,m\rangle,|\phi_{x,n}\rangle) = P_t(|0,m\rangle,U^*|x,n\rangle), \tag{A2}$$

which in turn gives

$$P_{t+1}(|0,m\rangle,|\phi\rangle) = qP_t(|0,m\rangle,U^*|\phi\rangle) + p\sum_{y,l}|\langle y,l|\phi\rangle|^2 P_{t+1}(|0,m\rangle,|y,l\rangle). \tag{A3}$$

This is our recursive formula. Also, for $t=1$, we have

$$P_1(|0,m\rangle,|\phi\rangle) = q|\langle\phi|U|0,m\rangle|^2 + p\sum_{y,l}|\langle y,l|\phi\rangle|^2 P_1(|0,m\rangle,U^*|y,l\rangle), \tag{A4}$$

and

$$P_1(|0,m\rangle,|x,n\rangle) = |\langle x,n|U|0,m\rangle|^2. \tag{A5}$$

By applying the recursive formulas (A3) and (A4) repeatedly, we have the following equation:

$$P_{m,n}(x,t) = P_t(|0,m\rangle,|x,n\rangle) = \sum_{s=1}^{t-1} pq^{s-1}\sum_{y,l}|\langle y,l|(U^*)^s|x,n\rangle|^2 P_{t-s}(|0,m\rangle,|y,l\rangle) + q^{t-1}|\langle x,n|U^t|0,m\rangle|^2. \tag{A6}$$

Note that by the definition of $W_{m,n}(x,t)$, we have that

$$|\langle y,l|(U^*)^s|x,n\rangle|^2 = |\langle x,n|U^s|y,l\rangle|^2 = W_{l,n}(x-y,s) \tag{A7}$$

and that

$$|\langle x,n|U^t|0,m\rangle|^2 = W_{m,n}(x,t). \tag{A8}$$

Therefore, Eq. (A6) becomes

$$P_{m,n}(x,t) = \sum_{s=1}^{t-1} pq^{s-1}\sum_{y,l} W_{l,n}(x-y,s)P_{m,l}(y,t-s) + q^{t-1}W_{m,n}(x,t). \quad (A9)$$

Now, by Eq. (A9), for $z\in\{z:|z|<\frac{1}{q}\}$,

$$P_{m,n}(x,z) = \delta_{x,n}^{0,m} + \frac{1}{p}Q_{m,n}(x,z) - Q_{m,n}(x,z) + \sum_{y,l} Q_{l,n}(x-y,z)P_{m,l}(y,z), \quad (A10)$$

where

$$\delta_{\beta,n}^{\alpha,m} = \begin{cases} 1, & \alpha=\beta, \ m=n, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, we take the Fourier transform on Eq. (A10) to obtain

$$\hat{P}_{m,n}(k,z) = \delta_n^m + \frac{q}{p}\hat{Q}_{m,n}(k,z) + \sum_l \hat{P}_{m,l}(k,z)\hat{Q}_{l,n}(k,z), \quad (A11)$$

where

$$\delta_n^m = \begin{cases} 1, & m=n, \\ 0, & \text{otherwise.} \end{cases}$$

The interchanges of summation are justified since the series absolutely converges. Now, denoting the matrices $[\hat{P}_{m,n}(k,z)]$ and $[\hat{Q}_{m,n}(k,z)]$ by P and Q, we have the following equation:

$$P = I + \frac{q}{p}Q + PQ, \quad (A12)$$

which is equivalent to

$$P(I-Q) = -\frac{q}{p}(I-Q) + \frac{1}{p}I. \quad (A13)$$

We complete the proof by the following lemma.

Lemma A.1. For $z\in\{z:|z|<1\}$, the matrix $I-Q$ is invertible.

Proof. For $z\in\{z:|z|<1\}$, if we let $Q_{m,n}=\hat{Q}_{m,n}(k,z)$, we have

$$\begin{aligned}|Q_{m,1}| + |Q_{m,2}| &= \frac{p}{q}\sum_{t=1}^{\infty} |\hat{W}_{m,1}(k,t)(qz)^t| + \frac{p}{q}\sum_{t=1}^{\infty} |\hat{W}_{m,2}(k,t)(qz)^t| \\ &< \frac{p}{q}\sum_{t=1}^{\infty} q^t[|\hat{W}_{m,1}(k,t)| + |\hat{W}_{m,2}(k,t)|] \\ &\le \frac{p}{q}\sum_{t=1}^{\infty} q^t[|\hat{W}_{m,1}(0,t)| + |\hat{W}_{m,2}(0,t)|] \le \frac{p}{q}\frac{q}{p} \\ &= 1, \quad \forall \ i. \end{aligned} \quad (A14)$$

Equation (A14) implies that $\|Q\|_\infty=\max_m\Sigma_n|Q_{m,n}|<1$. Therefore,

$$\|\sum_{j=0}^{\infty} Q^j\|_\infty \le \sum_{j=0}^{\infty} \|Q^j\|_\infty < \infty, \quad (A15)$$

i.e., the series $\Sigma_{j=0}^{\infty}Q^j$ converges. This implies that $(I-Q)^{-1}$ exists and

$$(I-Q)^{-1} = \sum_{j=0}^{\infty} Q^j.$$

By Lemma 1, $I-Q$ is invertible and together with Eq. (A13) we have

$$P = -\frac{q}{p}I + \frac{1}{p}(I-Q)^{-1}, \quad (A16)$$

which is exactly Eq. (21).

For $z\in\{z:|z|<\frac{1}{q}\}$, $|\det(I-Q)|<\infty$. Hence, $\det(I-Q)$ is analytic. Note also that

$$\hat{P}_{1,1}(k,z) = -\frac{q}{p} + \frac{1-Q_{2,2}}{p\det(I-Q)},$$

$$\hat{P}_{1,2}(k,z) = \frac{Q_{1,2}}{p\det(I-Q)},$$

$$\hat{P}_{2,1}(k,z) = \frac{Q_{2,1}}{p\det(I-Q)},$$

$$\hat{P}_{2,2}(k,z) = -\frac{q}{p} + \frac{1-Q_{1,1}}{p\det(I-Q)}. \quad (A17)$$

Therefore, the $\hat{P}_{m,n}(k,z)$ are meromorphic functions for $z\in\{z:|z|<\frac{1}{q}\}$.

APPENDIX B: PROOF OF THEOREM IV.1

To obtain formula (22), first we need to know the formulas of $\hat{W}_{m,n}(k,t)$, i.e., we look at the pure quantum walk in the Fourier transform.

Similar to the setup in [11], if we let the initial state be $|0,m\rangle$, and we let $\Psi_{m,n}(x,t)=\langle x,n|U^t|0,m\rangle$ be the coefficient of the walk at time t at coordinate $|x,n\rangle$, then $W_{m,n}(x,t)=|\Psi_{m,n}(x,t)|^2$. We also introduce $\hat{\Psi}_{m,n}(k,t)=\Sigma_x\Psi_{m,n}(x,t)e^{ikx}$ and $\hat{\Psi}_m(k,t)=[\hat{\Psi}_{m,1}(k,t),\hat{\Psi}_{m,2}(k,t)]^T$ in the Fourier transform as in [11]. The evolution operator in Fourier transform space, $U(k)$, is defined such that $\hat{\Psi}_m(k,t+1)=U(k)\hat{\Psi}_m(k,t)$. It is obtained in [11] that

$$U(k) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{ik} & e^{ik} \\ e^{-ik} & -e^{-ik} \end{pmatrix}. \quad (B1)$$

Therefore, if we let $A_k=\frac{1}{2}+\frac{\cos k}{2\sqrt{1+\cos^2 k}}$ and $C_k=\frac{e^{-ik}}{2\sqrt{1+\cos^2 k}}$, then for $t=2j-1$, we have

$$U^t(k) = \begin{pmatrix} -e^{-i\omega_k t} + 2A_k\cos\omega_k t & 2\bar{C}_k\cos\omega_k t \\ 2C_k\cos\omega_k t & e^{i\omega_k t} - 2A_k\cos\omega_k t \end{pmatrix}. \quad (B2)$$

Also, for $t=2j$, we have

$$U^t(k) = \begin{pmatrix} e^{-i\omega_k t} + 2A_k i \sin \omega_k t & 2i\bar{C}_k \sin \omega_k t \\ 2iC_k \sin \omega_k t & e^{i\omega_k t} - 2A_k i \sin \omega_k t \end{pmatrix}. \tag{B3}$$

Now, note that

$$\hat{\Psi}_{m,n}(k,0) = \sum_x \langle x,n|0,m\rangle e^{ikx} = \delta_n^m, \tag{B4}$$

and that $\hat{\Psi}_m(k,t)=[U(k)]^t\hat{\Psi}_m(k,0)$, we conclude that $\hat{\Psi}_{m,n}(k,t)=[U^t(k)]_{n,m}$.

Hence, for $t=2j-1$,

$$\hat{\Psi}_{1,1}(k,t) = -e^{-i\omega_k t} + 2A_k \cos \omega_k t,$$

$$\hat{\Psi}_{1,2}(k,t) = 2C_k \cos \omega_k t,$$

$$\hat{\Psi}_{2,1}(k,t) = 2\bar{C}_k \cos \omega_k t,$$

$$\hat{\Psi}_{2,2}(k,t) = e^{i\omega_k t} - 2A_k \cos \omega_k t. \tag{B5}$$

For $t=2j$,

$$\hat{\Psi}_{1,1}(k,t) = e^{-i\omega_k t} + 2A_k i \sin \omega_k t,$$

$$\hat{\Psi}_{1,2}(k,t) = 2iC_k \sin \omega_k t,$$

$$\hat{\Psi}_{2,1}(k,t) = 2i\bar{C}_k \sin \omega_k t,$$

$$\hat{\Psi}_{2,2}(k,t) = e^{i\omega_k t} - 2A_k i \sin \omega_k t. \tag{B6}$$

Since $W_{m,n}(x,t)=|\Psi_{m,n}(x,t)|^2$, in the Fourier transform,

$$\hat{W}_{m,n}(k,t) = \frac{1}{2\pi}\int_0^{2\pi} \hat{\Psi}_{m,n}(s,t)\hat{\Psi}_{m,n}(k-s,t)ds. \tag{B7}$$

We separate the real and imaginary parts of $\hat{W}_{m,n}(k,t)$ and get their formulas as follows. For $t=2j-1$,

$$\begin{aligned}\mathrm{Re}[\hat{W}_{1,1}(k,t)] &= \mathrm{Re}[\hat{W}_{2,2}(k,t)] \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\frac{\cos \omega_s t \cos \omega_{k-s} t}{\cos\omega_s \cos \omega_{k-s}} \cos s \cos(k-s) ds \\ &\quad - \frac{1}{2\pi}\int_0^{2\pi} \sin \omega_s t \sin \omega_{k-s} t ds,\end{aligned} \tag{B8}$$

$$\begin{aligned}\mathrm{Re}[\hat{W}_{1,2}(k,t)] &= \mathrm{Re}[\hat{W}_{2,1}(k,t)] \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos k}{2}\frac{\cos \omega_s t \cos \omega_{k-s} t}{\cos\omega_s \cos \omega_{k-s}} ds,\end{aligned} \tag{B9}$$

$$\begin{aligned}\mathrm{Im}[\hat{W}_{1,1}(k,t)] &= -\mathrm{Im}[\hat{W}_{2,2}(k,t)] \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{\sqrt{2}}\frac{\cos s}{\cos \omega_s} \cos \omega_s t \sin \omega_{k-s} t \right. \\ &\quad \left. + \frac{1}{\sqrt{2}}\frac{\cos(k-s)}{\cos \omega_{k-s}} \cos \omega_{k-s} t \sin \omega_s t \right) ds,\end{aligned} \tag{B10}$$

$$\begin{aligned}\mathrm{Im}[\hat{W}_{1,2}(k,t)] &= -\mathrm{Im}[\hat{W}_{2,1}(k,t)] \\ &= -\frac{1}{2\pi}\int_0^{2\pi} \frac{\sin k}{2}\frac{\cos \omega_s t \cos \omega_{k-s} t}{\cos \omega_s \cos \omega_{k-s}} ds.\end{aligned} \tag{B11}$$

For $t=2j$,

$$\begin{aligned}&\mathrm{Re}[\hat{W}_{1,1}(k,t)] \\ &\quad = \mathrm{Re}[\hat{W}_{2,2}(k,t)] \\ &\quad = -\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\frac{\sin \omega_s t \sin \omega_{k-s} t}{\cos \omega_s \cos \omega_{k-s}} \cos s \cos(k-s) ds \\ &\qquad + \frac{1}{2\pi}\int_0^{2\pi} \cos \omega_s t \cos \omega_{k-s} t ds,\end{aligned} \tag{B12}$$

$$\begin{aligned}\mathrm{Re}[\hat{W}_{1,2}(k,t)] &= \mathrm{Re}[\hat{W}_{2,1}(k,t)] \\ &= -\frac{1}{2\pi}\int_0^{2\pi} \frac{\cos k}{2}\frac{\sin \omega_s t \sin \omega_{k-s} t}{\cos \omega_s \cos \omega_{k-s}} ds,\end{aligned} \tag{B13}$$

$$\begin{aligned}\mathrm{Im}[\hat{W}_{1,1}(k,t)] &= -\mathrm{Im}[\hat{W}_{2,2}(k,t)] \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{\sqrt{2}}\frac{\cos s}{\cos \omega_s} \sin \omega_s t \cos \omega_{k-s} t \right. \\ &\quad \left. + \frac{1}{\sqrt{2}}\frac{\cos(k-s)}{\cos\omega_{k-s}} \sin \omega_{k-s} t \cos \omega_s t \right) ds,\end{aligned} \tag{B14}$$

$$\begin{aligned}\mathrm{Im}[\hat{W}_{1,2}(k,t)] &= -\mathrm{Im}[\hat{W}_{2,1}(k,t)] \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin k}{2}\frac{\sin \omega_s t \sin \omega_{k-s} t}{\cos \omega_s \cos \omega_{k-s}} ds.\end{aligned} \tag{B15}$$

Now we are ready to find the $\hat{P}_{m,n}(k,z)$ formula. We first introduce several short notations. We introduce the Σ_i's for $z\in\{z:|z|<\frac{1}{q}\}$. Let

$$\Sigma_1 = \mathrm{Re}(Q_{1,1}) = \frac{p}{q}\sum_{t=1}^{\infty} \{\mathrm{Re}[\hat{W}_{1,1}(k,t)]\}(qz)^t, \tag{B16}$$

$$\Sigma_2 = \mathrm{Re}(Q_{1,2}) = \frac{p}{q}\sum_{t=1}^{\infty}\{\mathrm{Re}[\hat{W}_{2,1}(k,t)]\}(qz)^t, \tag{B17}$$

$$\Sigma_3 = \mathrm{Im}(Q_{1,1}) = \frac{p}{q}\sum_{t=1}^{\infty}\{\mathrm{Im}[\hat{W}_{1,1}(k,t)]\}(qz)^t, \tag{B18}$$

$$\Sigma_4 = \mathrm{Im}(Q_{1,2}) = \frac{p}{q}\sum_{t=1}^{\infty}\{\mathrm{Im}[\hat{W}_{2,1}(k,t)]\}(qz)^t. \tag{B19}$$

Since $|\hat{W}_{m,n}(k,t)|\leq 1$, for $z\in\{z:|z|<\frac{1}{q}\}$, the above series all converge. Therefore, Σ_i's are all analytic in $\{z:|z|<\frac{1}{q}\}$.

Now $\det(I-Q)$ can be written as

$$\begin{aligned}\det(I-Q) &= 1 - Q_{1,1} - Q_{2,2} + Q_{1,1}Q_{2,2} - Q_{1,2}Q_{2,1}\\ &= (1-\Sigma_1)^2 - \Sigma_2^2 + \Sigma_3^2 - \Sigma_4^2.\end{aligned} \tag{B20}$$

Note that $\hat{P}(k,z)=\frac{1}{2}\Sigma_{m,n}\hat{P}_{m,n}(k,z)$. By the decoherence Eq. (21), this function can be written as

$$\begin{aligned}\hat{P}(k,z) &= -\frac{q}{p} + \frac{1}{2p}\frac{2 - Q_{1,1} + Q_{1,2} + Q_{2,1} - Q_{2,2}}{\det(I-Q)}\\ &= -\frac{q}{p} + \frac{1 - \Sigma_1 + \Sigma_2}{p[(1-\Sigma_1)^2 - \Sigma_2^2 + \Sigma_3^2 - \Sigma_4^2]}.\end{aligned} \tag{B21}$$

Therefore, once we have the formula of Σ_i's, we have the formula of $\hat{P}(k,z)$. To find Σ_i's formula, we first look for the formula for a real number $z\in(-\frac{1}{q},\frac{1}{q})$. Then we show that they are the desired formulas for all $z\in\{z:|z|<\frac{1}{q}\}$. Let

$$I_1 = \sum_{j=1}^{\infty}\cos[(2j-1)\omega_s]\cos[(2j-1)\omega_{k-s}](qz)^{2j-1}, \tag{B22}$$

$$I_2 = \sum_{j=1}^{\infty}\sin[(2j-1)\omega_s]\sin[(2j-1)\omega_{k-s}](qz)^{2j-1}, \tag{B23}$$

$$I_3 = \sum_{j=1}^{\infty}\cos[(2j)\omega_s]\cos[(2j)\omega_{k-s}](qz)^{2j}, \tag{B24}$$

$$I_4 = \sum_{j=1}^{\infty}\sin[(2j)\omega_s]\sin[(2j)\omega_{k-s}](qz)^{2j}, \tag{B25}$$

$$I_5 = \sum_{j=1}^{\infty}\cos[(2j-1)\omega_s]\sin[(2j-1)\omega_{k-s}](qz)^{2j-1}, \tag{B26}$$

$$I_6 = \sum_{j=1}^{\infty}\sin[(2j-1)\omega_s]\cos[(2j-1)\omega_{k-s}](qz)^{2j-1}, \tag{B27}$$

$$I_7 = \sum_{j=1}^{\infty}\sin[(2j)\omega_s]\cos[(2j)\omega_{k-s}](qz)^{2j}, \tag{B28}$$

$$I_8 = \sum_{j=1}^{\infty}\cos[(2j)\omega_s]\sin[(2j)\omega_{k-s}](qz)^{2j}. \tag{B29}$$

Since the Σ_i's are bounded, we can interchange the integral and the summation to write the Σ_i's as

$$\Sigma_1 = \frac{p}{q}\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2}\frac{\cos s\,\cos(k-s)}{\cos\,\omega_s\,\cos\,\omega_{k-s}}(I_1-I_4) - I_2 + I_3\right)ds, \tag{B30}$$

$$\Sigma_2 = \frac{p}{q}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\frac{\cos k}{\cos\,\omega_s\,\cos\,\omega_{k-s}}(I_1-I_4)ds, \tag{B31}$$

$$\Sigma_3 = \frac{p}{q}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\sqrt{2}}\left(\frac{\cos s}{\cos\,\omega_s}(I_5+I_7) + \frac{\cos(k-s)}{\cos\,\omega_{k-s}}(I_6+I_8)\right)ds, \tag{B32}$$

$$\Sigma_4 = \frac{p}{q}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\frac{\sin k}{\cos\,\omega_s\,\cos\,\omega_{k-s}}(-I_1+I_4)ds. \tag{B33}$$

Then, we have

$$I_1 - I_4 = \frac{1}{D}\cos\,\omega_s\,\cos\,\omega_{k-s}qz(1-q^2z^2), \tag{B34}$$

$$-I_2 + I_3 = \frac{1}{D}\left(-\frac{1}{2}\sin s\,\sin(k-s)qz + q^2z^2[\cos^2 s + \cos^2(k-s) - 1] - \frac{3}{2}\sin s\,\sin(k-s)q^3z^3 - q^4z^4\right), \tag{B35}$$

$$I_5 + I_7 = \frac{1}{D}\frac{1}{\sqrt{2}}qz\cos\,\omega_s[\sin(k-s) + 2qz\sin s + q^2z^2\sin(k-s)], \tag{B36}$$

$$I_6 + I_8 = \frac{1}{D}\frac{1}{\sqrt{2}}qz\cos\,\omega_{k-s}[\sin s + 2qz\sin(k-s) + q^2z^2\sin s], \tag{B37}$$

where

$$\begin{aligned}D &= \cos(k-2s)(q^3z^3 - 2\cos kq^2z^2 + qz)\\ &\quad + q^4z^4 - \cos kq^3z^3 - \cos kqz + 1.\end{aligned} \tag{B38}$$

Therefore,

$$\Sigma_1 = pz\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{D}\left(\cos(k-2s)qz(\cos k - qz) + \frac{1}{2}\cos k + \frac{1}{2}\cos kq^2z^2 - q^3z^3\right)ds, \tag{B39}$$

$$\Sigma_2 = \frac{1}{2}pz\cos k(1-q^2z^2)\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{D}ds, \tag{B40}$$

$$\Sigma_3 = pz\sin k\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{D}\left(\cos(k-2s)qz+\frac{1}{2}+\frac{1}{2}q^2z^2\right)ds, \tag{B41}$$

$$\Sigma_4 = \frac{1}{2}pz\sin k(1-q^2z^2)\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{D}ds. \tag{B42}$$

By the integral formula

$$\int\frac{dx}{b+c\cos ax} = \frac{2}{a\sqrt{b^2-c^2}}\arctan\left[\sqrt{\frac{b-c}{b+c}}\tan\left(\frac{1}{2}ax\right)\right] \tag{B43}$$

for $b>c$ and the fact that

$$q^4z^4 - \cos kq^3z^3 - \cos kqz + 1 > q^3z^3 - 2\cos kq^2z^2 + qz \tag{B44}$$

for $z\in(-\frac{1}{q},\frac{1}{q})$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{D}ds = ((1+qz)(1-qz)\{[q^2z^2-(1+\cos k)qz+1] \times[q^2z^2+(1+\cos k)qz+1]\}^{1/2})^{-1}. \tag{B45}$$

Letting

$$E = \sqrt{[q^2z^2-(1+\cos k)qz+1][q^2z^2+(1+\cos k)qz+1]}, \tag{B46}$$

we have

$$\Sigma_1 = \frac{pz}{q^2z^2-2\cos kqz+1}\left(\cos k - qz - \frac{\cos k - 2qz + \cos kq^2z^2}{2E}\right),$$

$$\Sigma_2 = pz\cos k\frac{1}{2E},$$

$$\Sigma_3 = \frac{pz\sin k}{q^2z^2-2\cos kqz+1}\left(1-\frac{1-q^2z^2}{2E}\right),$$

$$\Sigma_4 = -pz\sin k\frac{1}{2E}. \tag{B47}$$

Now that we have obtained the formulas of Σ_i's for $z\in(\frac{-1}{q},\frac{1}{q})$, we can check easily by taking the principal branch of the logarithm, that the formulas are analytic in $\{z:|z|<\frac{1}{q}\}$. Hence, by the analytic continuation theorem, they are the desired formulas for $z\in\{z:|z|<\frac{1}{q}\}$.

Finally, the theorem is obtained by applying Eq. (B47) to Eq. (B21).

APPENDIX C: OTHER PROOFS IN SECTION IV A

Proof of Theorem IV.2. Note that from the formula, for some $r<1$, we have

$$\mu_t = \frac{1}{i}\partial_k\hat{P}(0,t) = \frac{1}{2\pi i}\oint_{|z|=r}\frac{\partial_k\hat{P}(0,z)}{iz^{t+1}}dz = 0. \tag{C1}$$

The change of the order of integration and differentiation is justified since $\partial_k\hat{P}(k,z)$ is continuous on the contour.

Proof of Theorem IV.3. The denominator of $\hat{P}(k,z)$ has less than eight isolated roots. We shall now look for the root with the smallest absolute value. This root has no closed form. However, since we concentrate on the asymptotic behavior, we need only to know its behavior around $k=0$. The properties of this root are summarized in the following lemma.

Lemma C.1. Let $D(k,z)$ denote the denominator of $\hat{P}(k,z)$. Then the root of $D(k,z)=0$ in z, with $z=1$ when $k=0$ is of the smallest absolute value in a neighborhood of $k=0$. If we denote it by $z(k)$, then $z(k)$ has multiplicity one and can be written as follows:

$$z(k) = 1 + \partial_k z(0)k + o(k). \tag{C2}$$

Proof. For $k=0$, $D(0,z)=(1-z)(1-qz)[pz+(1-z)\sqrt{1+q^2z^2}]$. By solving this equation we can see that $z=1$ has the smallest absolute value. The root of the second smallest absolute value has a closed form expression, which can be found in the Appendix of [30]. We denote this root by $\tilde{z}(p)$. An expansion of the root around $p=0$ is

$$\tilde{z}(p) = 1 + \frac{\sqrt{2}}{2}p + \frac{1}{2}\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right)p^2 + o(p^2). \tag{C3}$$

Now, by continuity of k, $z(k)$ has the smallest absolute value in a neighborhood of $k=0$.

Since $\partial_z D(k,z)|_{k=0,z=1}\neq 0$, $z(k)$ has multiplicity one. We then apply the implicit function theorem to find its derivatives.

Remark C.1. For $p\to 1$, $D(0,z)\to 1-z$, for all $z\in\{z:|z|<\frac{1}{q}\}$, which implies that other roots go to infinity. In the limit, when $p=1$, there is only a single root.

Now we utilize the implicit function theorem to find $\partial_k z(0)$ and $\partial_k^2 z(0)$ where $z(k)$ is defined implicitly by $D(k,z)\equiv 0$.

Taking the first derivative we obtain

$$0 = \partial_k D(k,z) = -pq\sin kz^3 + pq\cos k3z^2\partial_k z - (pq+p)2z\partial_k z - p\sin kz + p\cos k\partial_k z + E(2z\partial_k z + 2\sin kz - 2\cos k\partial_k z) + \partial_k E(z^2 - 2\cos kz + 1). \tag{C4}$$

For $k=0$ and $z=1$, the equation becomes

$$(pq-p)\partial_k z = 0, \tag{C5}$$

which implies that

$$\partial_k z(0) = 0. \tag{C6}$$

Now, for $\partial_k^2 z(0)$, we can take the second derivative on $D(k,z)$ to obtain

$$
\begin{aligned}
0=\partial_k^2 D(k,z) &= -pq\cos kz^3-2pq\sin k3z^2\partial_k z\\
&\quad +pq\cos k[3z^2\partial_k^2 z+6z(\partial_k z)^2]-(pq+p)[2z\partial_k^2 z+2(\partial_k z)^2]\\
&\quad -p\cos kz-2p\sin k\partial_k z+p\cos k\partial_k^2 z+E(2z\partial_k^2 z\\
&\quad +2\cos kz-2\cos k\partial_k^2)+2\partial_k E(2z\partial_k z+2\sin k\partial_k z\\
&\quad -2\cos k\partial_k z)\partial_k^2 E(z^2-2\cos kz+1),
\end{aligned}
\tag{C7}
$$

which in turn gives

$$\partial_k^2 z(0)=\frac{p+2\sqrt{1+q^2}-2}{p}. \tag{C8}$$

Similarly, taking the third derivative of $D(k,z)=0$ gives

$$\partial_k^3 z(0)=0. \tag{C9}$$

Also, by taking the fourth derivative we obtain

$$
\begin{aligned}
\partial_k^4 z(0)=\frac{1}{p^3(1+q^2)^{1/2}}&[76q^4-83q^3(1+q^2)^{1/2}+16q^3\\
&+68q^2-q^2(1+q^2)^{1/2}-37q(1+q^2)^{1/2}\\
&+16q-23(1+q^2)^{1/2}+28].
\end{aligned}
\tag{C10}
$$

Hence we have the expansion of $z(k)$ at $k=0$,

$$z(k)=1+\frac{p+2\sqrt{1+q^2}-2}{2p}k^2+O(k^4). \tag{C11}$$

The residue of $\frac{\hat{P}(k,z)}{z^{t+1}}$ is

$$\mathrm{Res}\left(\frac{\hat{P}(k,z)}{z^{t+1}},z(k)\right)=\left(\frac{1}{z(k)}\right)^{t+1}\lim_{z\to z(k)}[z-z(k)]\hat{P}(k,z). \tag{C12}$$

We now prove another lemma.

Lemma C.2. We have

$$\lim_{z\to z(k)}[z(k)-z]\hat{P}(k,z)=1+O(k^2) \tag{C13}$$

as $k\to 0$.

Proof. Note that $\forall\ z\neq 1$,

$$\lim_{k\to 0}[z(k)-z]\hat{P}(k,z)=1, \tag{C14}$$

i.e., $\forall\ \epsilon>0$, $\exists\delta$, so that,

$$|[z(k)-z]\hat{P}(k,z)-1|<\epsilon \tag{C15}$$

for $|k|<\delta$. Equation (C15) implies that

$$\lim_{z\to z(k)}|[z(k)-z]\hat{P}(k,z)-1|\leq\epsilon \tag{C16}$$

for $|k|<\delta$. Hence,

$$|\lim_{z\to z(k)}[z(k)-z]\hat{P}(k,z)-1|\leq\epsilon \tag{C17}$$

for $|k|<\delta$, i.e.,

$$\lim_{k\to 0}\lim_{z\to z(k)}[z(k)-z]\hat{P}(k,z)=1. \tag{C18}$$

Now, for a small $r_1>0$, such that $z(k)$ is the only pole inside the circle $|z-1|=r_1$, we have

$$
\begin{aligned}
&\lim_{k\to 0}\lim_{z\to z(k)}\frac{1}{k}\{[z(k)-z]\hat{P}(k,z)-1\}\\
&\quad=\lim_{k\to 0}\frac{1}{k}\left(\frac{1}{2\pi i}\oint_{|z-1|=r_1}\hat{P}(k,z)dz-1\right)\\
&\quad=\frac{1}{2\pi i}\oint_{|z-1|=r_1}\partial_k\hat{P}(0,z)dz=0.
\end{aligned}
\tag{C19}
$$

Similarly, we also have

$$
\begin{aligned}
&\lim_{k\to 0}\lim_{z\to z(k)}\frac{1}{k^2}\{[z(k)-z]\hat{P}(k,z)-1\}\\
&=\frac{1}{2\pi i}\oint_{|z-1|=r_1}\partial_k^2\hat{P}(0,z)dz\\
&=\mathrm{Res}\left(\frac{z}{(1-z)^2}+\frac{2z^2(-1+\sqrt{1+q^2z^2})}{(1-z)^2[pz+(1-z)\sqrt{1+q^2z^2}]},1\right)\\
&=\frac{p+2\sqrt{1+q^2}-2}{p}-\frac{2q^2}{p\sqrt{1+q^2}}-\frac{2}{p^2}(1+q^2-\sqrt{1+q^2}).
\end{aligned}
\tag{C20}
$$

Therefore, $\lim_{z\to z(k)}[z(k)-z]\hat{P}(k,z)=1+O(k^2)$.

Now for any fixed $k\in[0,2\pi]$, the characteristic function of the distribution $P(\frac{x}{\sqrt{t}},t)$ is $\hat{P}(\frac{k}{\sqrt{t}},t)$. Since the roots of $D(k,z)$ are isolated, we can set $r(p)=1+\frac{\sqrt{2}}{2}p$ so that $|z(\frac{k}{\sqrt{t}})|<r(p)$ and other roots are outside the circle $\{|z|=r(p)\}$. Furthermore, when t is large, $\frac{k}{\sqrt{t}}$ is small, hence the lemmas are applicable. We define the contour C as $C=\{z:|z|=r_0\}\cup\{z:|z|=r(p)\}$, where $r_0<1$.

By definition,

$$\hat{P}\left(\frac{k}{\sqrt{t}},t\right)=\frac{1}{2\pi i}\oint_{|z|=r_0}\frac{\hat{P}\left(\frac{k}{\sqrt{t}},z\right)}{z^{t+1}}dz. \tag{C21}$$

Since $z(\frac{k}{\sqrt{t}})$ is the only pole inside the contour, we have

$$
\begin{aligned}
&-\mathrm{Res}\left(\frac{\hat{P}\left(\frac{k}{\sqrt{t}},z\right)}{z^{t+1}},z\left(\frac{k}{\sqrt{t}}\right)\right)\\
&\quad=\frac{1}{2\pi i}\left(\oint_{|z|=r_0}\frac{\hat{P}\left(\frac{k}{\sqrt{t}},z\right)}{z^{t+1}}dz-\oint_{|z|=r(p)}\frac{\hat{P}\left(\frac{k}{\sqrt{t}},z\right)}{z^{t+1}}dz\right).
\end{aligned}
\tag{C22}
$$

For fixed $0<p\leq 1$, $\sup_{k,|z|=r(p)}|\hat{P}(\frac{k}{\sqrt{t}},z)|<\infty$. Hence,

$$\oint_{|z|=r(p)} \frac{\hat{P}\left(\frac{k}{\sqrt{t}},z\right)}{z^{t+1}}dz = O[r(p)^{-t}]. \quad \text{(C23)}$$

We have

$$\hat{P}\left(\frac{k}{\sqrt{t}},t\right) = -\operatorname{Res}\left(\frac{\hat{P}\left(\frac{k}{\sqrt{t}},z\right)}{z^{t+1}},z\left(\frac{k}{\sqrt{t}}\right)\right) + O[r(p)^{-t}]. \quad \text{(C24)}$$

Note that by Eq. (C13), we have

$$\lim_{t\to\infty}\lim_{z\to z(k/\sqrt{t})}\left[z\left(\frac{k}{\sqrt{t}}\right)-z\right]\hat{P}\left(\frac{k}{\sqrt{t}},z\right) = 1 + O(t^{-1}). \quad \text{(C25)}$$

Note also that by Eq. (C11),

$$z\left(\frac{k}{\sqrt{t}}\right) = 1 + \frac{p+2\sqrt{1+q^2}-2}{2p}\frac{k^2}{t} + O(t^{-2}), \quad \text{(C26)}$$

which implies that

$$\begin{aligned}\left(\frac{1}{z\left(\frac{k}{\sqrt{t}}\right)}\right)^{t+1} &= \left(1+\frac{p+2\sqrt{1+q^2}-2}{2p}\frac{k^2}{t}+O(t^{-2})\right)^{-(t+1)}\\ &= \left(1-\frac{p+2\sqrt{1+q^2}-2}{2p}\frac{k^2}{t}+O(t^{-2})\right)^{t+1}\\ &= \exp\left(-\frac{p+2\sqrt{1+q^2}-2}{2p}k^2\right)+O(t^{-1}), \quad \forall\ k.\end{aligned} \quad \text{(C27)}$$

Therefore, by (C12), $\forall\ k\in[0,2\pi]$,

$$\hat{P}\left[z\left(\frac{k}{\sqrt{t}}\right),t\right] = \exp\left(-\frac{p+2\sqrt{1+q^2}-2}{2p}k^2\right)+O(t^{-1}), \quad \text{(C28)}$$

as $t\to\infty$. Hence, the limiting distribution of the symmetric decoherent Hadamard walk is Gaussian with variance $v=\frac{p+2\sqrt{1+q^2}-2}{p}$.

Proof of Theorem IV.4. For the symmetric walk, we can also find its long-term variance by the generating functions. Let C be the same contour as before, when t is large, $z=1$ is the closest root to 0 among all that of the denominator of $\hat{P}(k,z)$.

Note that

$$-\partial_k^2\hat{P}(0,z) = \frac{z}{(1-z)^2} + \frac{2z^2(-1+\sqrt{1+q^2z^2})}{(1-z)^2[pz+(1-z)\sqrt{1+q^2z^2}]}, \quad \text{(C29)}$$

and that

$$\begin{aligned}&\operatorname{Res}\left[\frac{1}{z^{t+1}}\left(\frac{z}{(1-z)^2}+\frac{2z^2(-1+\sqrt{1+q^2z^2})}{(1-z)^2[pz+(1-z)\sqrt{1+q^2z^2}]}\right),1\right]\\ &= \left(-1-\frac{2(\sqrt{1+q^2}-1)}{p}\right)t+\frac{2q^2}{p\sqrt{1+q^2}}\\ &\quad+\frac{2}{p^2}(1+q^2-\sqrt{1+q^2}).\end{aligned} \quad \text{(C30)}$$

Hence,

$$\begin{aligned}V(x,t) &= -\partial_k^2\hat{P}(0,t) = -\frac{1}{2\pi i}\oint_C\frac{\partial_k^2\hat{P}(0,z)}{z^{t+1}}dz\\ &= -\operatorname{Res}\left[\frac{1}{z^{t+1}}\left(\frac{z}{(1-z)^2}\right.\right.\\ &\quad\left.\left.+\frac{2z^2(-1+\sqrt{1+q^2z^2})}{(1-z)^2[pz+(1-z)\sqrt{1+q^2z^2}]}\right),1\right]+O[r(p)^{-t}]\\ &= \frac{p+2\sqrt{1+q^2}-2}{p}t-\frac{2q^2}{p\sqrt{1+q^2}}\\ &\quad-\frac{2}{p^2}(1+q^2-\sqrt{1+q^2})+O[r(p)^{-t}].\end{aligned} \quad \text{(C31)}$$

The change of the order of integration and differentiation is justified since $\partial_k^2\hat{P}(k,z)$ is continuous on the contour. Hence, the long-term variance of the walk is $\frac{p+2\sqrt{1+q^2}-2}{p}t-\frac{2q^2}{p\sqrt{1+q^2}}-\frac{2}{p^2}(1+q^2-\sqrt{1+q^2})+O[r(p)^{-t}]$.

APPENDIX D: PROOFS IN SECTION IVB

From the decoherence equation we have

$$\hat{P}_{1,1}(k,z) = -\frac{q}{p}+\frac{1}{p}\frac{1-Q_{2,2}}{\det(I-Q)}, \quad \text{(D1)}$$

$$\hat{P}_{1,2}(k,z) = \frac{1}{p}\frac{Q_{1,2}}{\det(I-Q)}. \quad \text{(D2)}$$

Let $\hat{\hat{P}}(k,z)=\hat{P}_{1,1}(k,z)+\hat{P}_{1,2}(k,z)$ be the generating function of the walk starting at $|0,1\rangle$. Then

$$\begin{aligned}\hat{\hat{P}}(k,z) &= -\frac{q}{p}+\frac{1-\Sigma_1+i\Sigma_3+\Sigma_2+i\Sigma_4}{p\det(I-Q)}\\ &= \hat{P}(k,z)+i\frac{\Sigma_3+\Sigma_4}{p\det(I-Q)}.\end{aligned} \quad \text{(D3)}$$

Note that Σ_3 and Σ_4 both have a factor of $\sin k$, we denote $\frac{\Sigma_3}{\sin k}$ and $\frac{\Sigma_4}{\sin k}$ by Σ_3 and Σ_4, respectively.

Proof of Theorem IV.5. Note that

$$\partial_k\hat{\tilde{P}}(0,z) = \partial_k\hat{P}(0,z) + \left. i\partial_k\left(\sin k\frac{\Sigma_3+\Sigma_4}{p\det(I-Q)}\right)\right|_{k=0}$$
$$= \left. i\left(\frac{\Sigma_3+\Sigma_4}{p}\right)\right|_{k=0}$$
$$= i\frac{z(\sqrt{1+q^2z^2}-1)}{(1-z)[pz+(1-z)\sqrt{1+q^2z^2}]}. \tag{D4}$$

Let C be the same contour as before. When t is large, $z=1$ is closest to 0 among the roots of the above denominator. Hence, for fixed p,

$$\tilde{\mu}_t = \frac{1}{i}\partial_k\hat{\tilde{P}}(0,t) = \frac{1}{2\pi i}\oint_C \frac{\partial_k\hat{\tilde{P}}(0,z)}{iz^{t+1}}dz$$
$$= \frac{1}{2\pi i}\oint_C \frac{(\sqrt{1+q^2z^2}-1)}{z^t(1-z)[pz+(1-z)\sqrt{1+q^2z^2}]}dz$$
$$= \mathrm{Res}\left(\frac{(\sqrt{1+q^2z^2}-1)}{z^t(1-z)(pz+(1-z)\sqrt{1+q^2z^2})},1\right) + O[r(p)^t]$$
$$= \frac{\sqrt{1+q^2}-1}{p} + O[r(p)^t]. \tag{D5}$$

Proof of Theorem IV.6. Note that $-\partial_k^2\hat{\tilde{P}}(0,z)$ must be real. Hence,

$$-\partial_k^2\hat{\tilde{P}}(0,z) = -\partial_k^2\hat{P}(0,z). \tag{D6}$$

Therefore, the formula is the same as before.

Proof of Theorem IV.7. We want to show that $P(\frac{x-\tilde{\mu}_t}{\sqrt{t}},t) \to N(0,v)$. The long-term characteristic function is

$$\hat{\tilde{P}}\left(\frac{k}{\sqrt{t}},t\right)e^{-i(\tilde{\mu}_t k/\sqrt{t})}$$
$$= \hat{P}\left(\frac{k}{\sqrt{t}},t\right)e^{-i(\tilde{\mu}_t k/\sqrt{t})}$$
$$+ e^{-i(\tilde{\mu}_t k/\sqrt{t})}\sin\frac{k}{\sqrt{t}}\frac{1}{2\pi i}\oint_C \frac{1}{z^{t+1}}\left(\frac{\Sigma_3+\Sigma_4}{p\det(I-Q)}\right)\left(\frac{k}{\sqrt{t}},z\right)dz$$

$$= \exp\left(-\frac{p+2\sqrt{1+q^2}-2}{2p}k^2\right) + O(t^{-1/2}). \tag{D7}$$

Hence, the limiting distribution of the decoherent Hadamard walk starting from 0 with coin state 1 is Gaussian with variance $v=\frac{p+2\sqrt{1+q^2}-2}{p}$.

APPENDIX E: ALTERNATIVE PROOF OF THEOREM 1

By Theorem IV.1,

$$-\partial_k^2\hat{P}(0,z) = \frac{z}{(1-z)^2} + \frac{2z^2(-1+\sqrt{1+q^2z^2})}{(1-z)^2[pz+(1-z)\sqrt{1+q^2z^2}]}. \tag{E1}$$

Note that the variance of the walk at time t is given by

$$V(x,t) = -\partial_k^2\hat{P}(0,t) \tag{E2}$$

and

$$\hat{P}(0,z) = \sum_t \hat{P}(0,t)z^t. \tag{E3}$$

Thus, $V(x,t)$ is the tth coefficient of the Taylor expansion of $-\partial_k^2\hat{P}(0,z)$.

As $p\to 0$, the symmetric decoherent Hadamard walk becomes the pure Hadamard walk. In particular, the function $V_0(z)=\Sigma_{t=1}V_Q(x,t)z^t$ can be obtained from Eq. (E1),

$$V_0(z) = \frac{z}{(1-z)^2} + \frac{2z^2}{(1-z)^3}\left(1-\frac{1}{\sqrt{1+z^2}}\right). \tag{E4}$$

Comparing the coefficients of the Taylor expansion of $V_0(z)$ gives

$$V_Q(x,t) = t - \sum_{j=1}^{[(t-2)/2]}(t-2j)(t-2j-1)(-1)^j\left(\frac{1}{2}\right)^{2j}\frac{1}{j}\frac{(2j)!}{(j!)^2}. \tag{E5}$$

As $t\to\infty$,

$$\frac{V_Q(x,t)}{t^2} \to -\sum_{j=1}^{\infty}(-1)^j\left(\frac{1}{2}\right)^{2j}\frac{1}{n}\frac{(2j)!}{(j!)^2} = 1-\frac{1}{\sqrt{2}}. \tag{E6}$$

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